

Convergence of quasi-Fuchsian groups using critical exponent

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Abstract

We prove that a sequence of quasi-Fuchsian representations for which the critical exponent converges to the topological dimension of the boundary of the group (larger than 2), converges up to subsequence and conjugacy to a totally geodesic representation.

1 Introduction

Given Γ a cocompact lattice of $\text{Isom}(\mathbb{H}^m)$ and a totally geodesic copy of \mathbb{H}^m into \mathbb{H}^n , $n > m$, we can see Γ as a discrete group of $\text{Isom}(\mathbb{H}^n)$. Indeed the isometry group of \mathbb{H}^m can naturally be seen as a subgroup of $\text{Isom}(\mathbb{H}^n)$ preserving the totally geodesic copy of $\mathbb{H}^m \subset \mathbb{H}^n$. We call this representation $\rho_0 : \Gamma \rightarrow \text{Isom}(\mathbb{H}^n)$ a *Fuchsian* representation. If one choose another copy of \mathbb{H}^m inside \mathbb{H}^n , the new Fuchsian representation is conjugated by an element of $\text{Isom}(\mathbb{H}^n)$ to ρ_0 . By Mostow rigidity, if $m \geq 3$, every representations of Γ inside $\text{Isom}(\mathbb{H}^m)$ are conjugated by an element of $\text{Isom}(\mathbb{H}^m)$. So all discrete, faithful and totally geodesic representations of Γ into $\text{Isom}(\mathbb{H}^n)$ are conjugated to ρ_0 , if $m \geq 3$. If $m = 2$, there exists non conjugate representations of Γ inside $\text{Isom}(\mathbb{H}^2)$, this is the Teichmüller space of Γ . We will suppose for the rest of the paper that $m \geq 3$.

Even if the representation ρ_0 cannot be deform inside $\text{Isom}(\mathbb{H}^m)$, there exists discrete and faithful deformations of ρ_0 in $\text{Isom}(\mathbb{H}^n)$ which are not anymore Fuchsian, ie. which does not preserve any totally geodesic copy of \mathbb{H}^m . We call them *quasi-Fuchsian* representations. There is a numerical invariant which measures how far from being Fuchsian a representation is; it is called the *critical exponent* and defined in the following way:

$$\delta(\rho(\Gamma)) := \limsup_{R \rightarrow +\infty} \frac{1}{R} \log \text{Card}\{\gamma \in \Gamma \mid d(\rho(\gamma)x, x) \leq R\},$$

it is independent of the base point $x \in \mathbb{H}^n$ thanks to the triangle inequality. It measure the exponential growth rate of an orbit inside \mathbb{H}^n .

By a simple computation using volume of balls, we can see that $\delta(\rho_0(\Gamma)) = m - 1$. In fact, a theorem of C. Yue, [Yue96] shows that critical exponent distinguishes Fuchsian representation: a quasi-Fuchsian representation ρ is conjugated to ρ_0 if and only if $\delta(\rho(\Gamma)) = m - 1$. Later, Besson-Courtois-Gallot, [BCG99, Theorem 1.14] showed that the convex-cocompact hypothesis is not needed, and proved that if ρ , a discrete and faithful representation of Γ , satisfies $\delta(\rho(\Gamma)) = m - 1$ then ρ is conjugated to ρ_0 . (In [BCG99] the theorem is cited with the convex-cocompact hypothesis, however they explained just after that the hypothesis is not needed).

Remark In dimension 2 the corresponding statement is $\delta(\rho(\Gamma))$ is equal to 1 if and only if it preserves a totally geodesic copy, but it is not necessarily conjugated to ρ_0 . However in this dimension, a lot of work has been done, and we know some examples where we can compute the limit of the critical exponent for a sequence of quasi-Fuchsian representations, see [McM99]. Moreover the work of A. Sanders, [San14b] shows that for a sequence of quasi-Fuchsian representations (if we suppose that the injectivity radius is bounded below) if the critical exponent goes to 1 then the sequence is close to a totally geodesic one. The aim of this article is to show a corresponding result in higher dimension, and we can even obtain convergence due to the absence of non trivial deformations inside $\text{Isom}(\mathbb{H}^m)$.

Theorem 1. *Let $m \geq 3$ and ρ_j be a sequence of quasi-Fuchsian representations. If $\delta(\rho_j(\Gamma)) \rightarrow m - 1$ then up to subsequence and conjugacy ρ_j converges to ρ_0 .*

Let us make some comments. Usually theorems often go in the opposite direction: we suppose that the sequence of groups converges (algebraically, geometrically or strongly) and give a result on the continuity of critical exponent. (Even the result of A. Sanders, does not show convergence.) For example, if one knows that ρ_j converges algebraically to a *convex cocompact* representation ρ_∞ , then a theorem of McMullen [McM99, Theorem 7.1], implies that $\delta(\rho_\infty(\Gamma)) = m - 1$ and hence by Yue's Theorem [Yue96], we know that ρ_∞ is conjugated to ρ_0 . The fact that we know the geometric structure of the limit representation is very important in the Theorem of McMullen. He explained in his paper how we can obtain sequence of representations for which the critical exponent is not continuous.

However in our case, putting together some deep theorems, we can show it is sufficient to prove that ρ_k converges algebraically to some representation ρ_∞ , for Theorem 1 to be true. Comparing to McMullen's work, here we do not need to know that the limit is convex cocompact, or that there is strong convergence.

Proposition 2. *Let ρ_k be a sequence of discrete and faithful representations in $\text{Isom}(\mathbb{H}^n)$, converging algebraically to ρ_∞ . Suppose that $\delta(\rho_k(\Gamma)) \rightarrow m - 1$ then $\delta(\rho_\infty(\Gamma)) = m - 1$ and therefore, ρ_∞ is conjugated to ρ_0 .*

Proof. First we use a theorem of Kapovich, [Kap08, Theorem 1.1], saying that if a sequence of discrete and faithful representations in $\text{Isom}(\mathbb{H}^n)$ converges, then the limit is discrete and faithful. Moreover a result of Bishop-Jones [BJ97] says that the critical exponent is lower semi-continuous, therefore

$$\delta(\rho_\infty) \leq \liminf \delta(\rho_k(\Gamma)) = m - 1.$$

We conclude by the Theorem of Besson-Courtois-Gallot previously cited, to conclude that ρ_∞ is conjugated to ρ_0 . \square

In their article Bishop-Jones give the result for subgroups of $\text{Isom}(\mathbb{H}^3)$, however their proof work in any dimension.

Therefore, the task is to show that under the critical exponent hypothesis the sequence of representations ρ_k converges algebraically to some representation. For this we will adapt the construction of Besson, Courtois, Gallot [BCG07].

2 The Besson, Courtois, Gallot construction

Let $Y = \mathbb{H}^m$ and $X = \mathbb{H}^n$. We are going to recall their construction of a sequence of maps $F_j : Y \rightarrow X$, $(\Gamma, \rho_j(\Gamma))$ -equivariant for which we can control the Jacobian and shows that it converges (up to subsequence and conjugation).

The maps F_j are the compositions of the following two:

- The first is the map $y \rightarrow \mu_y$ which goes from Y to the set of finite measures on ∂X . It associates to a point y the push forward of the Patterson-Sullivan measure (ν_y) on ∂Y by an equivariant homomorphism f_j from $\Lambda(\Gamma) = \partial Y$ to $\Lambda(\rho_j(\Gamma)) \subset \partial X$. We normalize μ_y into a probability measure.
- The second is the barycenter map going from the set of finite measures on ∂X to the space X . It associates to a measure μ , the unique point $\text{bar}(\mu)$, where the function:

$$\mathcal{B} : x \rightarrow \int_{\partial X} \beta_X(\xi, x) d\mu(\xi),$$

reaches its minimum. Here $\beta_X(\xi, x)$ is the Busemann function on X , normalized by taking an origin $o \in X$. It is shown in [BCG95] that $\text{bar}(\mu)$ is well defined as soon as μ has no atoms whose measure is greater than $\frac{1}{2}\mu(\partial X)$.

We define the map F_j by

$$F_j(y) := \text{bar}(\mu_y).$$

The Patterson-Sullivan density on Y , ν_y , satisfies, $\nu_{\gamma y} = \gamma_*(\nu_y)$ for all $\gamma \in \Gamma$. The barycenter map satisfies $\text{bar}(\gamma_*\mu) = \gamma(\text{bar}(\mu))$, for all $\gamma \in \text{Isom}(X)$. Therefore, the maps F_j are $(\Gamma, \rho_j(\Gamma))$ -equivariant.

Following [BCG99, BCG07], we introduce the quadratic forms, k_y, h_y and h'_y defined on $T_{F_j(y)}X$, $T_{F_j(y)}X$ and T_yY , by

$$\begin{aligned} k_{y,j}(v, w) &= \int_{\partial Y} Dd\beta_X|_{(F_j(y), f_j(\xi))}(v, w) d\nu_y(\xi) \\ h_{y,j}(v, w) &= \int_{\partial Y} d\beta_X|_{(F_j(y), f_j(\xi))}(v) d\beta_X|_{(F_j(y), f(\xi))}(w) d\nu_y(\xi) \\ h'_y(u, t) &= \int_{\partial Y} d\beta_Y|_{(y, \xi)}(u) d\beta_Y|_{(y, \xi)}(t) d\nu_y(\xi). \end{aligned}$$

for all $v, w \in T_{F_j(y)}X$ and $u, t \in T_yY$. We denote by $K_{y,j}, H_{y,j}$ and H'_y the corresponding symmetric endomorphisms. Note in particular that h' is independent of j and invariant by Γ , therefore, there exists $C > 0$ independent of y and j such that $\|H'\| \leq C$.

In order to prove that F_j converges, we will study the behavior of these quadratic forms. We list the principal properties that they satisfy:

Since ν_y is normalized into a probability and $\|d\beta_X\|_X = \|d\beta_Y\|_Y = 1$ we have:

$$\text{Tr}(H_{y,j}) \leq 1$$

$$\text{Tr}(H'_y) \leq 1.$$

The implicit functions theorem gives that F_j satisfies:

$$\int_{\partial Y} Dd\beta_X|_{(F_j(y), f(\xi))}(\cdot, d_y F_j(u)) d\nu_y(\xi) = (m-1) \int_{\partial Y} d\beta_X|_{(F_j(y), f(\xi))}(\cdot) d\beta_Y|_{(y, \xi)}(u) d\nu_y(\xi).$$

Now the Cauchy-Schwarz inequality applied on the second member of this equation gives:

$$k_{y,j}(v, dF_j(u)) \leq (m-1)h_{y,j}(v, v)^{1/2}h'_y(u, u)^{1/2}. \quad (1)$$

Definition 3. The p -Jacobian of a function $F : Y \rightarrow X$ is defined by

$$\text{Jac}_p F(y) = \sup \|dF_y(u_1) \wedge \cdots \wedge dF_y(u_p)\|,$$

where the supremum is taken over all p -orthonormal frames of $T_y^1 Y$.

When $p = m = \dim(Y)$ we will write $\text{Jac}F$.

By considering an orthonormal basis on $T_{F_j(y)} X$, it gives the following inequality on the determinants:

$$\det(\tilde{K}_{y,j}) \text{Jac}F_j(y) \leq (m-1)^m \det(\tilde{H}_{j,y})^{1/2} \det(H'_y)^{1/2}, \quad (2)$$

where $\tilde{K}_{y,j}$ and $\tilde{H}_{j,y}$ designed the restriction to $dF_j(T_y^1(Y)) \subset T_{F(y)}^1 X$ of $K_{y,j}$ and $H_{y,j}$. Since $\det(H'_y) \leq \left(\frac{1}{m} \text{Tr}(H'_y)\right)^m = \frac{1}{m^m}$, we have

$$\text{Jac}F_j(y) \leq \frac{(m-1)^m \det(\tilde{H}_{j,y})^{1/2}}{m^{m/2} \det(\tilde{K}_{y,j})} \quad (3)$$

Since X is the hyperbolic space of constant curvature, by direct computations we obtain that: $Dd\beta_X = g_X - d\beta_X \otimes d\beta_X$ therefore, $k_{y,j} = g_X - h_{y,j}$, and then

$$\det \tilde{K}_{y,j} = \det(\text{Id} - \tilde{H}_{j,y}) \quad (4)$$

We conclude as in [BCG99] that:

$$\text{Jac}F_j(y) \leq \frac{(m-1)^m \det(\tilde{H}_{j,y})^{1/2}}{m^{m/2} \det(\text{Id} - \tilde{H}_{j,y})} \quad (5)$$

Fact The map $H \rightarrow \frac{\det(H)^{1/2}}{\det(\text{Id} - H)}$ defined on positive definite symmetric matrices of dimension $m \geq 3$ and trace less than 1 achieves its unique maximum on $H = \frac{1}{m} \text{Id}$. The value of this maximum is $\frac{m^{m/2}}{(m-1)^m}$. Therefore we have:

$$\text{Jac}F_j(y) \leq \left(\frac{(m-1)}{m-1}\right)^m = 1 \quad (6)$$

Thanks to the previous fact, Besson-Courtois-Gallot, proved in [BCG95]:

Lemma 4. [BCG95] If $\text{Jac}F_j(y) = 1$ then

$$\frac{\det \tilde{H}_{j,y}^{1/2}}{\det(\text{Id} - \tilde{H}_{j,y})} = \frac{m^{m/2}}{(m-1)^m} \quad \text{and} \quad \tilde{H}_{j,y} = \frac{1}{m} \text{Id}.$$

The following approximation is clear:

Lemma 5. $\forall \eta > 0, \exists \epsilon_0 > 0, \forall 0 < \epsilon \leq \epsilon_0$, if $|\text{Jac}F_j - 1| \leq \epsilon$ then

$$\left| \frac{(\det \tilde{H}_{y,j})^{1/2}}{\det(\text{Id} - \tilde{H}_{y,j})} - \frac{m^{m/2}}{(m-1)^m} \right| \leq \eta$$

We can also obtain an approximation of the second part of Lemma 4. Indeed their proof shows that there is no maximum of $\frac{(\det H)^{1/2}}{\det(\text{Id} - H)}$ on the boundary $\{\text{Sym}_p^{++}, \cap \text{Tr}(H) = 1\}$. This implies the following approximation:

Lemma 6. $\forall \eta > 0, \exists \epsilon_0 > 0, \forall 0 < \epsilon \leq \epsilon_0$, if $|\text{Jac}F_j - 1| \leq \epsilon$ then

$$\left\| \tilde{H}_{y,j} - \frac{1}{m} \text{Id} \right\| \leq \eta.$$

3 Proof of Theorem 1

We now show that up to subsequence and conjugacy F_j converges. We follow the main steps presented in the paragraph 4 of [BCG07].

Step 1 : Almost everywhere convergences of $\tilde{H}_{y,j}$.

Lemma 7. *Up to subsequence, $\text{Jac}(F_j(y))$ converges almost everywhere to 1.*

Proof. Applying the same construction and inequalities in the direction $X \rightarrow Y$ we get a sequence of $(\rho_j(\Gamma), \Gamma)$ -invariant maps, G_j satisfying, for all $p \in [3, n]$

$$\text{Jac}_p G_j(x) \leq \frac{\delta(\rho_j(\Gamma))^p}{(p-1)^p} \quad (7)$$

The maps $H_j = G_j \circ F_j : Y \rightarrow Y$ are Γ -invariant, of degree 1 and satisfies $\text{Jac}(H_j(y)) = \text{Jac}_m(H_j(y)) \leq \left(\frac{\delta(\rho_j(\Gamma))}{m-1}\right)^m$. Therefore:

$$\text{Vol}(Y/\Gamma) = \int_{Y/\Gamma} \text{Jac}_m(H_j(y)) dy \leq \left(\frac{\delta(\rho_j(\Gamma))}{m-1}\right)^m \text{Vol}(Y/\Gamma).$$

Since $\delta(\rho_j(\Gamma)) \xrightarrow{j \rightarrow \infty} m-1$, this implies that up to subsequence, $\text{Jac}_m(H_j(y))$ converges almost everywhere to 1. Let $x = F_j(y)$, since $\text{Jac}_m(H_j(y)) \leq \text{Jac}_m(G_j(x))\text{Jac}(F_j(y))$ it implies that $\text{Jac}(F_j) \rightarrow 1$ almost everywhere. \square

We will still denote this converging subsequence by the index j .

Using Lemmas 5 and 6, the previous result shows:

Lemma 8. *For almost every $y \in Y$, $\lim_{j \rightarrow \infty} \tilde{H}_{y,j} = \frac{1}{m} \text{Id}$.*

As in [BCG07], we will denote by \tilde{H}_0 the quadratic form on $dF_j(T_y^1(Y)) \subset T_{F_j(y)}^1 X$ equal to $\frac{1}{m} \text{Id}$.

Step 2 : Uniform convergence of $\tilde{H}_{y,j}$ to \tilde{H}_0 . We denote by μ_j the largest eigenvalue of $\tilde{H}_{y,j}$.

Lemma 9. [BCG07, Lemma 4.7] *Let y, y' be two points in Y such that $\mu_j \leq 1 - \frac{1}{m}$ on every points of the geodesic from y to y' , then there exists a constant C such that*

$$d_X(F_j(y), F_j(y')) \leq C d_Y(y, y').$$

Since our setting is a bit simpler than the one in [BCG95, BCG07], we make a proof without the technical problems that appears therein.

Proof. Recall Inequality (1)

$$g_0(K_{y,j} dF_j(y)u, v) \leq (m-1)h_{y,j}(v, v)^{1/2} h'_y(u, u)^{1/2}$$

We already remarked that $\|H'_{y,j}\|$ is bounded independently of $j \in \mathbb{N}$ and $y \in Y$. Therefore there exists $C_1 > 0$ such that

$$g_0(K_{y,j} dF_j(y)u, v) \leq C_1 h_{y,j}(v, v)^{1/2} \leq C_1 \sqrt{\mu_j(y)}.$$

Moreover, by hypothesis we have $\tilde{K}_{y,j} = \text{Id} - \tilde{H}_{y,j} \geq (1 - \mu_j(y)) \text{Id}$. Therefore, by taking $v = \frac{dF_j(y)}{\|dF_j(y)\|}$, we have

$$(1 - \mu_j(y)) \|dF_j(y)\|_{g_0} \leq C_1 \sqrt{\mu_j(y)}$$

Let $\alpha(t)$ be the geodesic joining y and y' . The last inequality, implies that there exists C independent of $j \in \mathbb{N}$ and $u \in T_{\alpha(t)}^1 Y$ such that, $\|dF_j(u)\|_{g_0} \leq C$. The lemma follows thanks to the mean value inequality. \square

The following lemma can also be found in [BCG95, BCG07] with a function F_j slightly more complicated.

Lemma 10. [BCG07] *With the same notations as in the previous lemma, let P denotes the parallel transport from $F_j(y)$ to $F_j(y')$ along the geodesic in X joining these two points. We have*

$$\|h_{y,j} - h_{y',k} \circ P\| \leq 4d_X(F_j(y), F_j(y')).$$

Proof. Let $\beta(t)$ be the geodesic from $F_j(y)$ to $F_j(y')$. Let Z be a unit parallel vector field along β , and called $Z_1 = Z(F_j(y))$ and $Z_2 = Z(F_j(y'))$. We then have:

$$h_{y',k}(Z_2, Z_2) - h_{y,j}(Z_1, Z_1) = \int_{\partial X} (d\beta_X|_{(F_j(y'), f_j(\xi))}(Z_2))^2 d\xi - \int_{\partial X} (d\beta_X|_{(F_j(y), f_j(\xi))}(Z_1))^2 d\xi$$

On one hand, we have:

$$\left| d\beta_X|_{(F_j(y'), f_j(\xi))}(Z_2) - d\beta_X|_{(F_j(y), f_j(\xi))}(Z_1) \right| \leq \left(\sup_t \left| Dd\beta_X|_{\beta(t)}(\dot{\beta}, Z) \right| \right) d_X(F_j(y), F_j(y')).$$

Since Z is unitary, $Dd\beta_x = g_X - d\beta_X \otimes d\beta_X$, and $\|d\beta_X|_{(F_j(y'), f_j(\xi))}(\cdot)\| \leq 1$ we have

$$\left(\sup_t \left| Dd\beta_X|_{\beta(t)}(\dot{\beta}, Z) \right| \right) \leq 2.$$

On the other hand, for any unitary vector u, v we have:

$$\left| d\beta_X|_{x,\xi}(u) + d\beta_X|_{x,\xi}(v) \right| \leq 2.$$

Therefore,

$$\left| (d\beta_X|_{(F_j(y'), f_j(\xi))}(Z_2))^2 - (d\beta_X|_{(F_j(y), f_j(\xi))}(Z_1))^2 \right| \leq 4d_X(F_j(y), F_j(y')).$$

\square

Lemma 11. [BCG07, Lemma 4.9] $\tilde{H}_{y,j}$ converges uniformly to \tilde{H}_0 as $j \rightarrow \infty$.

The proof is exactly the same as in [BCG07, Lemme 4.9]. It uses Egoroff's theorem, Lemmas 9 and 10, but does not use the particular form of F_j .

Step 3 : Uniform convergence of F_j .

The following lemma, corresponds to Lemma 4.10 in [BCG07].

Lemma 12. *Up to subsequence and composition by an element of $\text{Isom}(X)$, F_j converges uniformly to a continuous map $F : Y \rightarrow X$.*

Proof. For all $\epsilon > 0$, there exists $j \gg 0$ such that for all $y \in Y$ we have

$$\tilde{H}_{y,j} \leq \tilde{H}_0 + \epsilon \text{Id} \quad \text{and} \quad \tilde{K}_{y,j} \geq \tilde{K}_0 - \epsilon \text{Id}.$$

Therefore, using Inequality (1) there exists $C > 0$ independent of $j \in \mathbb{N}$ such that, for all $u \in T^1Y$:

$$\|dF_j(u)\|_X \leq C.$$

This means that the sequence F_j is C -Lipschitz. We fix y_0 in Y and $x_0 \in X$. Let $\gamma_j \in \text{Isom}(\mathbb{H}^m)$ be an element such that $\gamma_j F_j(y_0) = x_0$, and call $F'_j = \gamma_j \circ F_j$. Since γ_j is an isometry of X we have $\|dF'_j(u)\|_X \leq C$. Now for any point $y \in Y$ we have

$$d_X(F'_j(y), F'_j(y_0)) \leq C d_Y(y, y_0).$$

Since $F'_j(y_0)$ is chosen to be equal to x_0 , $\{F'_j(y)\}_{j \in \mathbb{N}}$ is bounded, and we conclude by Ascoli's theorem. \square

Let F be a uniform limit of F'_j .

Lemma 13. *The sequence $\rho_j : \Gamma \rightarrow \text{Isom}(\mathbb{H}^n)$ admits a subsequence which converge algebraically to a discrete and faithful representation ρ_∞ .*

Proof. For every $\gamma \in \Gamma$, the sequence $\rho_j(\gamma)$ is equicontinuous since they are 1-Lipschitz maps. Let $j \gg 0$ such that for all $y \in Y$, $d(F_j(y), F(y)) \leq \epsilon$. Now, take any point $x \in X$

$$\begin{aligned} d(\rho_j(\gamma)x, x) &\leq d(\rho_j(\gamma)x, \rho_j(\gamma)F(y)) + d(\rho_j(\gamma)F(y), F(\gamma y)) + d(F(\gamma y), x) \\ &\leq d(x, F(y)) + d(F(\gamma y), x) + d(\rho_j(\gamma)F(y), \rho_j(\gamma)F_j(y)) + d(\rho_j(\gamma)F_j(y), F(\gamma y)) \\ &\leq d(x, F(y)) + d(F(\gamma y), x) + d(F(y), F_j(y)) + d(F_j(\gamma y), F(\gamma y)) \\ &\leq d(x, F(y)) + d(F(\gamma y), x) + 2\epsilon. \end{aligned}$$

Therefore $\{\rho_j(\gamma)x \mid k \in \mathbb{N}\}$ is relatively compact for all $x \in X$ and all $\gamma \in \Gamma$. Ascoli's Theorem asserts that $\rho_j(\gamma)$ admits a converging subsequence, call it $\rho_\infty(\gamma)$. We can make a diagonal argument on a finite set of generators to find a subsequence still denoted ρ_j for which all $\rho_j(\gamma_i)$ converges to some $\rho_\infty(\gamma_i)$, where $\Gamma = \langle \gamma_i, i \in [1, r] \rangle$. By definition this means that ρ_j converges to a representation ρ_∞ .

Now we use the result cited in the introduction due to Kapovich [Kap08, Theorem 1.1] which implies that ρ_∞ is discrete and faithful. \square

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